arrangement yields

$$\sum_{i=1}^{M} D_i E_i \sum_{i=1}^{M} z_i \leq \sum_{i=1}^{M} D_i \sum_{i=1}^{M} z_i E_i \qquad (28)$$

which can be rewritten as

$$\sum_{i=1}^{M} \left(\frac{\partial z_i}{\partial t}\right) E_i \sum_{i=1}^{M} z_i - \sum_{i=1}^{M} \left(\frac{\partial z_i}{\partial t}\right) \sum_{i=1}^{M} z_i E_i$$

$$= \sum_{i=1}^{M} z_i \sum_{i=1}^{M} \left(\frac{\partial z_i}{\partial t}\right) (E_i - E') \geqq 0 \qquad (29)$$

Comparison of Equations (29) and (16) indicates that in this case

$$\left(\frac{\partial E'}{\partial t}\right)_T \geqq 0 \qquad Q.E.D. \qquad (30)$$

Similarly, it can be shown that if

$$E_i \geqq E_{i+1} \quad \text{for} \quad i = 1, 2, \ldots, M-1 \qquad (31)$$

then

$$\left(\frac{\partial E'}{\partial t}\right)_T \leqq 0$$

When $f(k_j, t)$ vanishes for some of the species inequality (26) becomes an equality since in this case $z_j = D_j = 0$. However, this does not affect the proof.

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Optimal Feedback Control of a Class of Linear Tubular Processes

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Optimal output feedback control law requiring measurement of the output only has been considered for a class of tubular processes. The advantage of this formulation, which requires a single measurement of the output only, over a previous result which requires an infinite number of measurements, has been demonstrated through a heat exchanger example. However, when the terminal time is greater than the residence time of fluid, a time-delay element is needed.

Processes involving transport of fluids or solids have transportation lags associated with trem, or they can be approximated by models with time delays. Frequently, the control of the output (at the exit) of these tubular processes can be described by differential-difference equations. Open-loop controls of processes described by differential-difference equations have been considered by a number of investigators (*1* to *4*). Koppel (*1*) considered the time-optimal control of processes described by a differential-difference equation. Ray (*2*) treated the optimal open-loop control of processes with pure time delay using a variational approach. Seinfeld and Lapidus (*3*) considered the open-loop control of a distributed-parameter process described by differential-difference equations. Using a direct programming approach Lim (*4*) presented and solved time optimal control problems.

For control systems governed by linear differential-difference equations, Koepcke (*5*) developed a synthesis technique particularly suited for a digital computer. The optimum feedback control of a class of processes described by differential-difference equations was considered by Koppel et al. (*6*) as a special case of more general distributed-parameter process control, and they obtained near optimal feedback solution which, when applied to a heat exchanger, requires the measurement of the entire temperature profile along the length (hence requires an infinite number of sensors). More recently, Shih (*7*) considered a special case of the terminal time being less than one, $t_f < 1$. However, in this case the process reduces to the usual lumped-parameter process and truly distributed nature of the distributed-parameter processes appears only when $t_f > 1$. We present here a different approach and show that for the output transfer from one steady state to another the optimal control law requiring the measurement of the output alone, can be obtained for all time, that is, both cases $t_f \leqq 1$ and $t_f > 1$.

GENERAL CONTROL PROBLEM

Many tubular processes can be represented by differential-difference Equations (1) to (3), for example,

$$\sum_{i=0}^{n} a_i \, d^i y(t)/dt^i = k_1 \, m(t) - k_2 \, m(t - \tau)$$

where τ is the time delay. As shown by Lim (*4*) it is convenient to represent the input-output relationship by state and output equations of the type $x(t) = A\,x(t) + b\,m(t)$ and $y(t) = k_1 x_1(t) - k_2 x_2(t - \tau) S(t - \tau)$. Therefore, we consider a general class of processes with multiple inputs, multiple outputs, and a constant time delay in the state vector

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad (1)$$

$$x(0) = x_0 \tag{2}$$

$$y(t) = Cx(t) + D[x(t-a) - x_0]S(t-a) + E(t)x_0 \tag{3}$$

where $x(t)$ is the n-component state vector, $y(t)$ is the n-component output vector, $u(t)$ is the m-component control vector, A, C, and D are constant $n \times n$ matrices, B is an $n \times m$ constant matrix, E is an $n \times n$ time-variant matrix, $S(t-a)$ is a delayed unit step function, and a is a scalar constant. Controllability and observability are assumed. The type of processes that can be represented by Equations (1), (2), and (3) are, for example, a heat exchanger with wall temperature control (*1, 4*), or the flow-through heating of a thin sheet metal in a furnace by manipulating uniform furnace temperature (*8*), a heat exchanger with wall flux control (*1*) and a class of chemical reactors (*1*). They also represent a double pipe heat exchanger with steam heating (*3*) with $t_f \leq a$, and any parallel network of distributed-parameter processes characterized by pure time delay with or without lumped-parameter processes. Processes with pure time delay can be also represented by Equations (1), (2), and (3) with $C = 0$.

The problem is to find for the process of Equations (1) through (3) the optimal feedback control $u[x(t)]$ which minimizes the integral performance index of the quadratic type

$$J(u) = \frac{1}{2}\int_0^{t_f} (y^T Q y + u^T R u)\,dt \tag{4}$$

where Q is a constant, positive semidefinite, symmetric $n \times n$ matrix and R is a constant, positive-definite, symmetric $m \times m$ matrix, t_f is fixed, and $u(t)$ is not constrained.

Necessary Conditions for Optimality

The necessary conditions for optimality for a process with a fixed time delay in the state vector $x(t)$ and not in the control, have been derived by Kharatishvili (*9*). The optimal control is obtained by differentiating the Hamiltonian

$$H = \frac{1}{2}(y^T Q\, y + u^T R\, u) + p^T(Ax + Bu) \tag{5}$$

so that

$$u(t) = -R^{-1} B^T p(t) \tag{6}$$

where the adjoint vector $p(t)$ is continuous, piecewise differentiable, and given by

$$\begin{aligned}\dot{p}(t) &= -\frac{\partial H}{\partial x(t)^T} - \frac{\partial H}{\partial x(t-a)^T}\Bigg|_{t+a} \\ &= -A^T p(t) - C^T Q\{C\,x(t) + D[x(t-a) - x_0] \\ &\quad S(t-a) + E(t)\,x_0\} - D^T Q\{C\,x(t+a) \\ &\quad + D[x(t) - x_0]\,S(t) + E(t+a)\,x_0\}\,S(t) \\ &\qquad t \in [0, t_f - a]\end{aligned} \tag{7}$$

$$\begin{aligned}\dot{p}(t) &= -\frac{\partial H}{\partial x(t)^T} \\ &= -A^T p(t) - C^T Q\{C\,x(t) + D[x(t-a) \\ &\quad - x_0]\,S(t-a) + E(t)\,x_0\}\ t \in [t_f - a, t_f]\end{aligned} \tag{8}$$

subject to a terminal condition

$$p(t_f) = 0 \tag{9}$$

To obtain the optimal control one must solve the adjoint equations, Equations (7) and (8), with the terminal condition given by Equation (9). However, these adjoint equations are coupled with the state and thus Equations (1) and (2) must be solved together with Equations (7), (8), and (9). Thus, we now have the usual two-point boundary value problem in optimization.

Linear Feedback Control Law

For practical interest it is desirable to obtain the feedback control law. Following the procedure analogous to that used by Kalman (*10*) in ordinary lumped-parameter processes, a proper form in which to seek a solution is

$$p(t) = K(t)\ x(t) \qquad t \in [0, t_f] \tag{10}$$

Substitution of Equation (10) into Equations (6) and (1) yields

$$u(t) = -R^{-1} B^T K(t)\ x(t) \tag{11}$$

and

$$\dot{x}(t) = [A - B\,R^{-1} B^T K]\ x(t) \tag{12}$$

Substitution of Equation (11) into Equation (1) and integration of the resulting differential equation yield

$$x(t) = \phi(t, 0)\ x_0 \tag{13}$$

where

$$\phi(t_2, t_1) = \exp\left[\int_{t_1}^{t_2} (A - B\,R^{-1} B^T K)\ dt\right] \tag{14}$$

Differentiation of Equation (10) yields

$$\dot{p}(t) = \dot{K}(t)\ x(t) + K(t)\ \dot{x}(t) \tag{15}$$

Substitution of Equation (12) into Equation (15) and equating to Equations (7) and (8) yields the desired gain equations, Riccati-type equations.

$$\begin{aligned}&\dot{K} + KA + A^T K - KBR^{-1} B^T K = \\ &\quad -C^T QC - D^T QD\ S(t) - \{C^T QD[\phi(t-a, 0) \\ &\quad - I]\ S(t-a) + C^T QE(t) - D^T Q(D - E(t+a)) \\ &\quad S(t) + D^T QC\phi(t+a, 0)\}\ \phi^{-1}(t, 0)\ \ t \in [0, t_f - a]\end{aligned} \tag{16}$$

and

$$\begin{aligned}&\dot{K} + KA + A^T K - KBR^{-1} B^T K = \\ &\quad -C^T QC - C^T Q\{D[\phi(t-a, 0) - I]\ S(t-a) \\ &\quad + E(t)\}\phi^{-1}(t, 0)\ \ t \in [t_f - a, t_f]\end{aligned} \tag{17}$$

The boundary condition is

$$K(t_f) = 0 \tag{18}$$

The optimal feedback gain which is obtained through Riccati-type equations is independent of the initial conditions on the state and therefore can be precomputed entirely. When $D = E = 0$, the original system reduces to an ordinary lumped-parameter system and Equations (16) and (17) reduce as they should to a single familiar Riccati equation for lumped-parameter systems

$$\dot{K} + KA + A^T K - KBR^{-1} B^T K = -C^T QC \tag{19}$$

As a special case, when the terminal time is less than or equal to the residence time of fluid $t_f \leq a$ or equivalently $D = 0$, the process given by Equations (1), (2), and (3) reduces to a lumped-parameter process, $\dot{x} = Ax + Bu$ and $y = Cx + Ex_0$. In this case one of the gain equations, Equation (16), vanishes and the other, Equation (17), reduces to

$$\begin{aligned}&\dot{K} + KA + A^T K - KBR^{-1} B^T K \\ &\quad + C^T Q\ E(t)\ \phi^{-1}(t, 0) = -C^T QC\end{aligned} \tag{20}$$

This limiting case has been considered by Shih (*7*). Note also that for the usual lumped-parameter process with

$E(t) = 0$, Equation (20) reduces to a familiar Riccati equation, Equation (19).

Example

The optimal control theory developed above is applied to obtain optimal control of the tubular plug flow heat exchanger by manipulation of the spatially uniform wall temperature. This problem has been considered by Koppel et al. (*6*). The system is given by

$$\frac{\partial T}{\partial t} + \frac{\partial T}{\partial r} = P[u(t) - T] \tag{21}$$

$$T(0, t) = 0 \tag{22}$$

$$T(r, 0) = u(0) \ [1 - e^{-Pr}] \tag{23}$$

Where $T(r, t)$ is a normalized deviation fluid temperature around a new desired steady state at normalized time t and normalized distance r, P is the ratio of heat exchange to heat capacity, and $u(t)$ is a normalized deviation wall temperature. Equations (21), (22), and (23) represent also the process of heating a moving slab through a furnace. In this case the furnace gas temperature may serve as $u(t)$. The objective is to determine how the normalized wall temperature be changed in a feedback fashion so that the outlet temperature $T(1, t)$ is driven from an initial undesired steady state to a new steady state minimizing a quadratic performance index

$$J[u(t)] = \frac{1}{2} \int_0^{t_f} [T^2(1, t) + \alpha u^2(t)] \, dt \tag{24}$$

where t_f is fixed, $u(t)$ is free and α is a constant. An approximate solution to this problem requiring an infinite number of sensors was obtained by Koppel et al. (*6*) by modifying the performance index Equation (24). Shih (*7*) considered this problem for a limited case where $t_f \leqq 1$. We present here a complete solution for all time, that is, $t_f < 1$ and $t_f \geqq 1$.

Laplace transformation of Equations (21), (22), and (23) yields

$$T(1, s) = \frac{P[1 - e^{-(s+P)}]}{s + P} [u(s) + u(0)/P] - e^{-P} u(0) (1 - e^{-s})/s \tag{25}$$

By extending a direct programming technique (*4, 11*) to Equation (25), the state and output equations are obtained as

$$\frac{dx(t)}{dt} + Px(t) = P \, u(t) \tag{26}$$

$$x(0) = u(0) \tag{27}$$

and

$$T(1, t) \overset{\Delta}{=} y(t) = x(t) - e^{-P} x(t - 1) \ S(t - 1) - e^{-P} x(0) \ S(1 - t) \tag{28}$$

The performance index becomes

$$J[u(t)] = \frac{1}{2} \int_0^{t_f} (y^2 + \alpha u^2) \ dt \tag{29}$$

From the equations of this system and of the general control system, we obtain

$$-A = B = P, \quad D = E = -\exp(-P), \quad C = Q = 1, \quad R = \alpha, \quad a = 1$$

and the Riccati-type equations, Equations (16) and (17) are

$$\dot{K} - 2PK - P^2K^2/\alpha = -1 - e^{-2P} + e^{-P} \{[\phi(t - 1, 0) - 1] \ S(t - 1) + 1 + \phi(t + 1, 0)\}\phi^{-1}(t, 0) \quad t \, \epsilon[0, t_f - 1] \tag{30}$$

$$\dot{K} - 2PK - P^2K^2/\alpha = -1 + e^{-P} \{[\phi(t - 1, 0) - 1] \ S(t - 1) + 1\}\phi^{-1}(t, 0) \quad t \, \epsilon[t_f - 1, t_f] \tag{31}$$

where

$$\phi(t, 0) = \exp\left[\int_0^t (-P - P^2K/\alpha) d\tau \right] \tag{32}$$

and

$$K(t_f) = 0 \tag{33}$$

There are two cases of interest: $t_f \leqq 1$ and $t_f > 1$. When the terminal time is less than or equal to one residence time of fluid $t_f \leqq 1$ the gain equation for the first time interval Equation (30) vanishes and the gain equation for the second time interval Equation (31) reduces to

$$\dot{K} - 2PK - P^2K^2/\alpha = -1 + \phi^{-1}(t, 0), \quad t_f \leqq 1, \quad t \, \epsilon[0, t_f] \tag{34}$$

Note that this gain equation and the optimal control $u(t) = -\frac{P}{\alpha} K(t) \ x(t)$ appear apparently different from the results given by Shih (*7*). Actually they are equivalent; the apparent differences are due to the fact that state representations are not unique so that the gain equations are not unique. However, the optimal control $u(t)$ must be the same.

The solutions of Equations (30) through (33) and

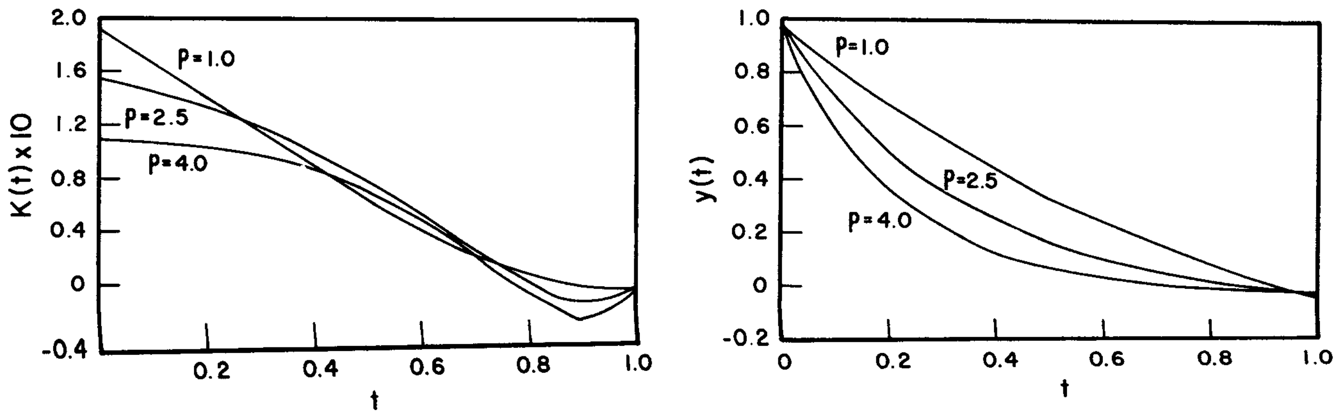

Fig. 1. Optimal gain and exit temperature $t_f = 1$, $\alpha = 2$.

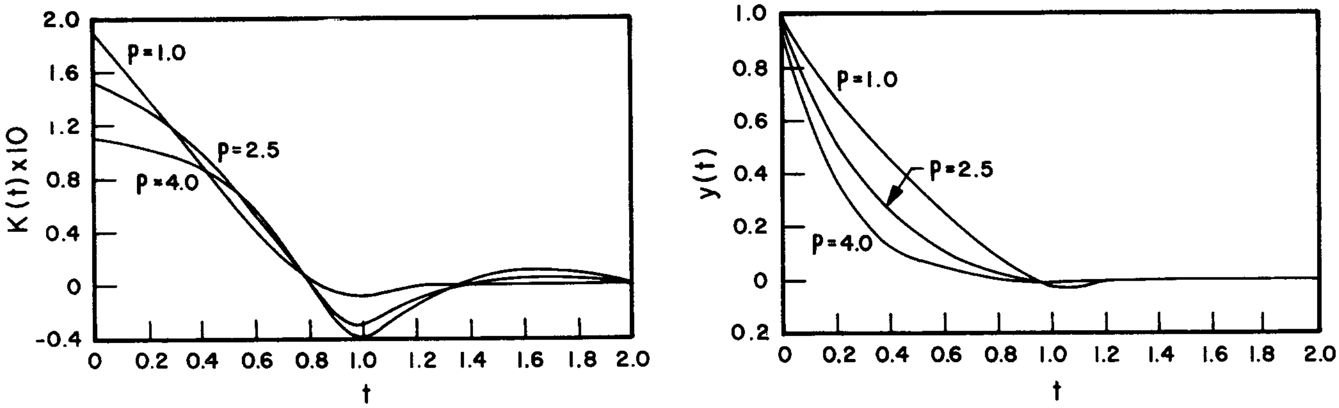

Fig. 2. Optimal gain and exit temperature $t_f = 2, \alpha = 2$.

Equations (32) through (34) give the optimal gains for $t_f > 1$, respectively. However, these equations are integro-differential equations and some iteration technique must be used for solution. For instance, for $t_f > 1$ the following iteration method is feasible:

1. Assume $K(t)$, say $K_a(t) = 0$ initially, or $K_a(t) = K_c(t)$ as calculated in 3.
2. Compute $\phi(t, 0)$ from Equation (32).
3. Compute $K_c(t)$ by integrating Equation (30) and (31).
4. Compare the computed and assume K's.

This process is repeated until $K_c^{-1}(t)|K_c(t) - K_a(t)|$ is less than a preassigned value, say 10^{-5}, for all t and the boundary condition of Equation (33) is met.

On the average about 10 iterations were needed to converge to the ultimate optimal gain, requiring less than 10 sec. of computer time on a CDC 6500 computer. The iteration was also initiated from different starting points and in each case it converged to the same gain. Computational results, obtained by this method, are shown in Figures 1 and 2 for $t_f = 1$ and $t_f = 2$ respectively. We note that for the case $t_f = 2.0$ the optimal gain $K(t)$ for $t > 1$ is nearly zero, indicating that nearly all control efforts are made while the fluid remains in the heat exchanger, $t < 1$. Notice also that the gain is negative around $t = 1$. This is due to the nature of the process, that is, the fluid going through the heat exchanger receives more and more energy as it approaches the exit and in fact some energy has to be removed in order to have the exit temperature come close to the desired value $[T(1, t_f) = 0]$. This is done by means of negative gain. The effect of the parameter P is as anticipated; the higher the ratio of heat exchange to heat capacity the faster and better the exit temperature response.

In analogy to lumped-parameter processes it is tempting to assume that the optimal gain is constant for a large terminal time $t_f = \infty$, that is, $K =$ constant for $t \in [1, \infty]$. When $t_f = \infty$, Equation (30) reduces to

$$\dot{K} - 2PK - P^2K^2/\alpha = -(1 + e^{-P}) + e^{-P}[1 + \phi(t + 1, 0)]\,\phi^{-1}(t, 0) \quad t \in [0, 1] \tag{35}$$

$$\dot{K} - 2PK - P^2K^2/\alpha = -(1 + e^{-P}) + e^{-P}[\phi(t - 1, 0) + \phi(t + 1, 0)]\phi^{-1}(t, 0) \quad t \in [1, \infty) \tag{36}$$

It is not difficult to show that for $t \in [1, \infty]$ the only non-negative constant K which satisfies Equation (36) is $K = 0$ and that the assumption of constant K for the second time interval $t \in [1, \infty)$ necessarily implies nonconstant $K(t)$ for $t \in [0, 1]$. The authors are not able to prove the first assumption. A similar result was observed by Khatri (*13*) for linear processes with a pure dead time in the control variable alone. Furthermore, in Figure 3 the comparison of our work with $K(t)$ for $t \in [0, 1]$ and $K = 0$ for $t \in [1, \infty)$ which is based on the measurement of the exit temperature alone, with the work of Shih (*12*) which is based on the measurement of fluid temperature along the entire length of the heat exchanger shows a complete agreement and seems to bear this out.

REALIZATION OF OPTIMAL FEEDBACK CONTROL

The optimal control law Equation (11), is a feedback control based on the state. When the state is accessible to measurement, the engineering realization of feedback control law presents no particular problem. For instance, analog elements may be used to generate $K(t)$ and to carry out multiplication $K(t)\ x(t)$. On the other hand when the state is not measurable it must be computed from the measurable output. In the above example the output is the exit fluid temperature, which is readily measurable, and the state is related to the output by Equation (28). The state then must be computed from the output through Equation (3) in general or Equation (28) for the example.

$$x(t) = C^{-1}\{y(t) - S(t - 1)\ D[x(t - 1) - (C + E)^{-1}y(0)] - E(C + D)^{-1}y(0)\}$$

or

$$x(t) = y(t) + e^{-P}[x(t - 1) - (1 - e^{-P})^{-1}y(0)]$$

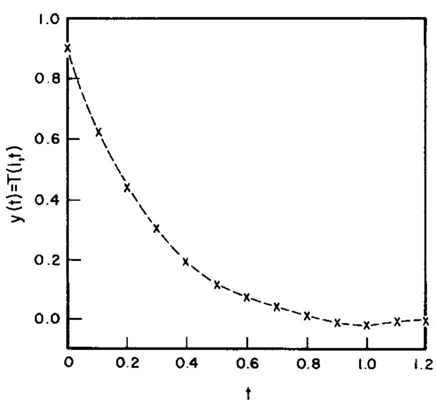

Fig. 3. Comparison of our results with Shih's (*12*) —— our results, $t_f = \infty$ $p = 2.5$, $\alpha = 0.5$
xxx Shih's results (*12*).

$$+ e^{-P} (1 - e^{-P})^{-1} y(0) \quad (37)$$

since observability of the system is assumed. Equation (37) shows that a time-delay element is necessary to generate $x(t-1)$, which is required for $x(t)$. For the case $t_f \leq 1$, Equation (37) reduces to

$$x(t) = y(t) + e^{-P} (1 - e^{-P})^{-1} y(0) \quad (38)$$

and no time-delay element is necessary. When $t_f > 1$, a time-delay element is necessary. A number of approximation methods and exact realizations are possible. The most common approximations are Padé approximations and some mechanical devices such as a recording tape or drum with a read head at some distance away from the writing head (*14*) and, of course, a digital or hybrid computer. Engineering realization of the optimal feedback control system with inaccessible state is given in Figure 4 for the example problem.

DISCUSSION

Optimal feedback control law for a class of linear processes described by a vector differential-difference equation has been presented. When the state is not available for measurement, a time delay element is necessary to construct the state from the measurement of the output.

When applied to a tubular heat exchanger the optimal control law requires only the measurement of the exit temperature (hence requires only one sensor), as opposed to the previously reported suboptimal control law (*6*), which requires the measurement of the fluid temperature over the entire length of the heat exchange (hence requires an infinite number of sensors). However, when the terminal time is greater than one residence time of the fluid, a time delay operator is needed. Whenever a time-delay operation can be readily realized the advantages of the optimal control law requiring a single measurement of the output over those requiring an infinite number of measurements are rather obvious.

When the final time is infinite the optimal gain matrix does not reduce to a constant matrix. The present work suggests that the optimal gain is time-varying for the first time interval equivalent in length to the residence time of fluid $t \in [0, a]$ and constant for the remaining time interval $t \in [a, \infty)$. However, the authors are not able to prove this.

Finally, the approach used here, when applied to the process with pure time delay, leads to Smith predictor control (*15*). In fact, one can rigorously show that for processes with pure dead time Smith predictor control is the optimal control based on the state rather than the output.

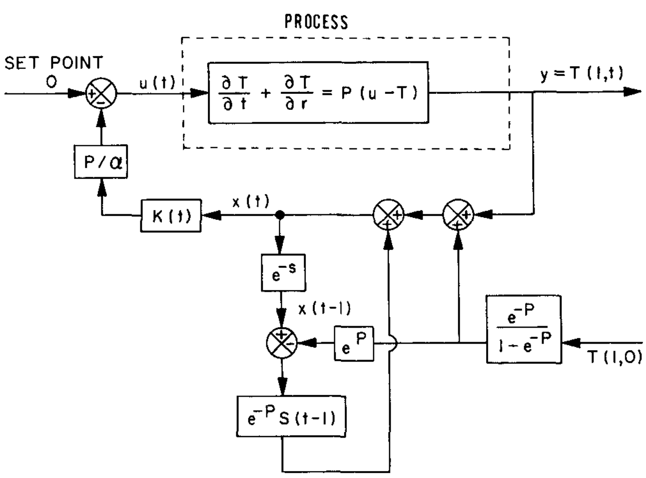

Fig. 4. Feedback control realization.

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NOTATION

A	=	$n \times n$ constant matrix
a	=	scalar constant
B	=	$n \times m$ constant matrix
C	=	$n \times n$ constant matrix
D	=	$n \times n$ constant matrix
$E(t)$	=	$n \times n$ matrix
k, k_1, k_2	=	scalar constants
$K(t)$	=	gain
P	=	ratio of heat exchange to heat capacity
$p(t)$	=	adjoint vector
Q	=	$n \times n$ constant symmetric matrix
R	=	$m \times m$ constant, positive definite, symmetric matrix
$S(t-1)$	=	delayed unit step function
s	=	Laplace transform variable
$T(1, s)$	=	Laplace transform of $T(1, t)$
$T(1, t)$	=	$T(r, t)$ at $r = 1$
$T(r, t)$	=	normalized fluid temperature, deviation from a new desired steady state
$U(s)$	=	Laplace transform of $u(t)$
$u(t)$	=	control vector
$x(t)$	=	state vector
x_0	=	state vector evaluated at $t = 0$
y	=	output

Greek Letters

α	=	weighting factor in performance index
τ	=	dead time

Superscript

T	=	transpose

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